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# Multiple re-entrant phenomenon on decorated square lattice Ising models

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**Abstract.** The re-entrant phenomenon is studied in the antiferromagnetic Ising model on a square lattice with the bonds decorated with two and three-coupled spin $\frac{1}{2}$ variables (dimer and trimer) in two different scenarios. In the first one ferromagnetic competing decorated bonds are anneal diluted in the system. The exact phase diagrams of the critical temperature as a function of the concentration and of the competition parameter are obtained. In contrast to the single spin bond decoration, a double re-entrant behaviour is observed. The bond percolation threshold $P_c = (1 - 1/\sqrt{2})/2$ is achieved and it is independent of the decorating variable. In the second scenario the pure axial decorated system is studied. The analytical temperature against competition parameter phase diagram and the specific heat are calculated and discussed. In this latter case multiple re-entrances are observed for a certain range of the competition parameter. The nature of this re-entrant phenomenon is discussed.

## 1. Introduction

Re-entrant phase transitions have been observed in a variety of physical systems, such as superconductors [1, 2], liquid crystals [3, 4], spin glasses [5] and adsorbed layers [6]. Re-entrant phases have also appeared in the phase diagrams of different model Hamiltonians. This phenomenon is characterised by the reappearance of a less ordered phase, following a more ordered one, as the temperature is lowered. For each physical system there is a distinct mechanism determining this reversal behaviour. In superconductors containing magnetic impurities, such as $La_{1-x}Th_x$ (Ce as impurity) a continuous transition from a magnetic system (LaCe) to a nearly non-magnetic system (ThCe) is observed with a re-entrant behaviour for low concentrations of Th [1]. In granular superconductors a mean field theory [2] predicts that the lowering of the resistance below the transition temperature is followed by a rise at lower temperatures, although

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in a recent calculation based on a self-consistent phase-phonon approximation this behaviour has not been found [7]. In liquid crystals it has been observed [3] that under high pressures the nematic phase reappears for temperatures lower than that of the smectic phase. More remarkably, a sequence of quadruple re-entrances has been found experimentally [4] in liquid crystals. Theoretical phase diagrams similar to the experimental one were obtained from calculations based on a frustrated spin-glass model [5]. Low field DC magnetisation measurements on the amorphous (FeMn) PBAl alloy [6] show that two second-order phase transitions occur with the magnetisation going to zero at two temperatures. An x-ray diffraction study of krypton on graphite [8] shows a modulated re-entrant fluid phase separating the commensurate-incommensurate phases, connecting continuously onto the high temperature fluid phase. The helical Potts lattice-gas model has been used to explain the pressure-temperature phase diagram of krypton on graphite [8]. A re-entrant spin-glass phase has also been observed in the high temperature superconductor compound $La_{2-x}(Sr, Ba)_xCuO_4$ for very small values of $x$ in the insulator phase [9] and a re-entrant behaviour of the superconducting phase has been predicted if one is to rely on the existence of antiferromagnetic underlying order to provide the pairing mechanism of the superconducting carriers [10]. The spin-glass re-entrant phases are explained by a frustration and disorder mechanism in model Hamiltonians. Theoretical interest in the re-entrance phenomenon has also received considerable attention. Re-entrant phase diagrams in a spin-one Ising model with biquadratic interactions were obtained by renormalisation group [11] and Monte Carlo methods [12]. It is explained by the competition between the ferromagnetic and the negative biquadratic coupling. A similar phenomenon has been found in the random bond Ising model [13] and in the BCC nearest-neighbour Ising antiferromagnet [14]. In order to explain spin-glass behaviour in iron–aluminium alloys, a spin $\frac{1}{2}$ competing Ising model has been studied [15] and a re-entrance is obtained. Also an amorphous Ising antiferromagnetic model [16] shows a re-entrance as a result of the competition. A decorated Ising model with competing interactions also shows re-entrant antiferromagnetic [17] and re-entrant ferromagnetic phases [17, 18]. An in-plane antiferromagnet in a magnetic field exhibits a re-entrant second-order phase boundary as shown by an extended mean field theory [19]. The re-entrant behaviour shown in these theoretical models differs in one aspect. Some of these models are pure, i.e., there is no disorder in the coupling constants [11–13], while others are disordered [14–18]. However the competition between the coupling interactions is essential for the appearance of a re-entrant phase. For the disordered competing systems, when frustration is present, the role of the entropy in determining the minimum of the free energy can account for this phenomenon [18].

In this paper we study the antiferromagnetic square lattice Ising model decorated with complex unities of Ising spins as bond decoration. Our main interest is to understand the mechanisms that produce the multiple re-entrant phenomenon. It has been shown recently [17] that the present model randomly decorated with a single ferromagnetic D-vector-bond spin has a single re-entrant phase arising from the local diluted competing effects due to the decoration. Here we consider in addition to the single case ($D=1$ of [17]) two species of complex unities of $\frac{1}{2}$ Ising spins as bond decoration: the dimer, a pair of antiferromagnetic coupled spins interacting ferromagnetically with site Ising spins, and the trimer, a set of three frustrated coupled spins interacting ferromagnetically with the site spins. In figure 1($a$) the single, dimer and trimer decoration are shown schematically. We note that there is no coupling between decorating bond spins belonging to distinct bonds. Therefore one is able to evaluate

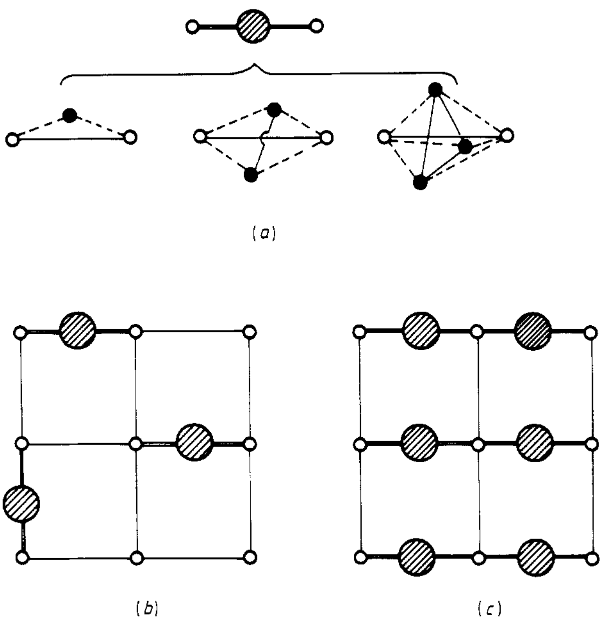


**Figure 1.** (*a*) Scheme for the monomer, dimer and trimer bond decoration. (*b*) Portion of random-decorated square lattice. (*c*) Portion of pure axial-decorated square lattice.

exactly the effective ferromagnetic interaction produced by the decoration [17, 20]. For certain values of the coupling constants this effective interaction which is also temperature dependent, can be compared with the original homogeneous antiferromagnetic one and competing effects come into play in the system. Moreover the additional frustrated degrees of freedom present in the dimer and trimer decorated bonds should play an important role for the entropic term of the free energy in order to produce the multiple re-entrant phenomenon. To distinguish between the re-entrant phenomenon produced by the increasing of the entropy due to disorder and the one produced by the increasing of the entropy due to the presence of additional genuine degrees of freedom we consider the present model in two scenarios: one with isotropically random annealed decorated bonds and one with pure axial decorated bonds. As we will see later on the latter shows a more pronounced multiple re-entrant phenomenon. In section 2, we study the competing random decorated model. We obtain the phase diagrams ($T_c x p$) and ($T_c x \alpha$), where $T_c$, $p$ and $\alpha$ stand for the transition temperature, the concentration of the decorated bonds and the competing coupling parameter, respectively. In section 3, the phase diagram ($T_c x \alpha$) and the specific heat are obtained for the pure axial decorated model. These results are exact and show a multiple re-entrance behaviour. In section 4, the main results are summarised.

## 2. Competing random decorated model

### 2.1. Model Hamiltonian

We consider an antiferromagnetic Ising model in a square lattice with nearest-neighbour interaction, $-\alpha J$, decorated with Ising spin classical variables, as described by the

Hamiltonian $H = \Sigma_b H_b$, with the bond Hamiltonian $H_b$ given by

$$H_b = \alpha J\sigma_1\sigma_2 - t_b\left(\sum_{\langle S,\sigma\rangle} JS_i(\lambda\sigma_1 + \lambda'\sigma_2) + \sum_{i=1}^{M} J_iS_iS_{i+1} + \mu\right) \tag{1}$$

where $t_b$ is the occupation parameter for the decorated-bond spins $S_i$, i.e., $t_b = 1$ (0) if the bond $b$ is decorated (undecorated) and $\mu$ is the chemical potential related to the annealing dilution. The decorating variables $S_i$ interact among themselves, with an exchange interaction $J_i = \gamma_i J$, forming an open or closed ($S_{i+M} = S_i$) finite chain of $M$ spins. The $S_i$ variables couple independently with the site variables $\sigma_1$ and $\sigma_2$ via the coupling constants $\lambda J$ and $\lambda' J$ where $\lambda$ and $\lambda'$ are arbitrary parameters. In figure 1($a$), we show the scheme of the decoration bonds and in figure 1($b$) a portion of the random decorated lattice.

By performing the sum over the bond variables $\{S_i, t_b\}$ using the method of the decoration transformation [20], we obtain for the partition function,

$$Z_N = A^{2N}(K_i, \eta)Z_0(K_{\text{eff}}) \tag{2}$$

where $K_i = \beta J_i$, $\eta = \exp(\beta\mu)$. $K_{\text{eff}} = \beta J_{\text{eff}}$ and $A(K_i, \eta)$ are given respectively by

$$\begin{aligned} K_{\text{eff}} &= K + \tfrac{1}{2}\ln[(1+\eta X)/(1+\eta Y)] \\ A(K_i, \eta) &= [(1+\eta X)(1+\eta Y)]^{1/2}. \end{aligned} \tag{3}$$

Here $X$ and $Y$ are given by the expressions appearing in table 1 for $M = 1$-3 bond decorating variables and $K = \beta J$. $Z_0(K_{\text{eff}})$ is the partition function of the two-dimensional Ising model on a square lattice with the effective temperature-dependent interaction given by expression (3). The fugacity associated with the bond occupation variable $t_b$ can be eliminated by imposing the annealed condition that the thermal average of $t_b$ should be equal to the concentration $p$ of decorated bonds, i.e.,

$$\langle t_b\rangle = \lim_{N\to\infty} (1/2N)(\partial/\partial(\beta\mu)) \ln Z = p. \tag{4}$$

After a direct calculation, we obtain for the unwanted fugacity the result,

$$\eta = [-S + (S^2 + 4pR)^{1/2}]/2R \tag{5}$$

**Table 1.** Analytical expressions for $X$ and $Y$ obtained from the partial summation over the bond variables for the monomer ($M = 1$), dimer ($M = 2$) and trimer ($M = 3$).

| | | |
|---|---|---|
| $M = 1$ | $X$ | $2\cosh(2K)$ |
| | $Y$ | $2$ |
| $M = 2$ | $X$ | $2e^{\gamma K}\cosh[2K(1+\lambda)] + 2e^{-\gamma K}$ |
| | $Y$ | $2e^{-\gamma K}\cosh[2K(1-\lambda)] + 2e^{+\gamma K}$ |
| $M = 3$ | $X$ | $2\{\exp[(\gamma_1+\gamma_2+\gamma_3)K]\cosh[K(2+2\lambda+\lambda_1+\lambda_2)] + \exp[(-\gamma_1-\gamma_2+\gamma_3)K$ $\times\cosh[K(2+2\lambda-\lambda_1-\lambda_2)] + \exp(\gamma_1-\gamma_2-\gamma_3)K]\cosh[K(\lambda_1+\lambda_2)]$ $+\exp[(\gamma_1+\gamma_2-\gamma_3)K]\cosh[K\lambda_1+\lambda_2)]\}$ |
| | $Y$ | $2\{\exp[(\gamma_1+\gamma_2+\gamma_3)K]\cosh[K(\lambda_1-\lambda_2)] + \exp[(-\gamma_1-\gamma_2+\gamma_3)K]$ $\times\cosh[K(\lambda_1-\lambda_2)] + \exp[(\gamma_1-\gamma_2-\gamma_3)K]\cosh[K(2-2\lambda+\lambda_1-\lambda_2)]$ $+\exp[(-\gamma_1+\gamma_2-\gamma_3)K]\cosh[K(2\lambda-2+\lambda_1-\lambda_2)]\}$ |

where

$$S = \tfrac{1}{2}(1+\varepsilon-2p)X + \tfrac{1}{2}(1-\varepsilon-2p)Y$$
$$R = XY(1-p)$$

and

$$\varepsilon = \langle\sigma_1\sigma_2\rangle = \lim_{N\to\infty} (1/2N)(\partial/\partial K_{\mathrm{eff}}) \ln Z(K_{\mathrm{eff}}) \tag{6}$$

is the nearest-neighbour-pair correlation function. The transition temperature is given by Onsager's result

$$\sinh 2K_{\mathrm{eff}} = \pm 1. \tag{7}$$

The + (−) sign holds for the ferromagnetic (antiferromagnetic) critical temperature. The phase diagram of the critical temperature as a function of the concentration of the decorating bonds can be analytically obtained from the previous equation. Eliminating $\eta$ from (7) with help of (5), we obtain the phase diagram $T_c xp$,

$$p = \{BXY + \tfrac{1}{2}[(X+Y) + \varepsilon(X-Y)]\}/(B^{-1} + BXY + X + Y) \tag{8}$$

where $B = (C-1)/(X-CY)$ and $C = \exp(2\alpha K)[\sqrt{2}+1]$.

The phase diagram of the critical temperature against the competing parameter $\alpha$ which measures the ratio between the intersite antiferromagnetic coupling and the ferromagnetic one between site and bond spins ($\lambda = \lambda' = 1$) is obtained by inverting (7), that is

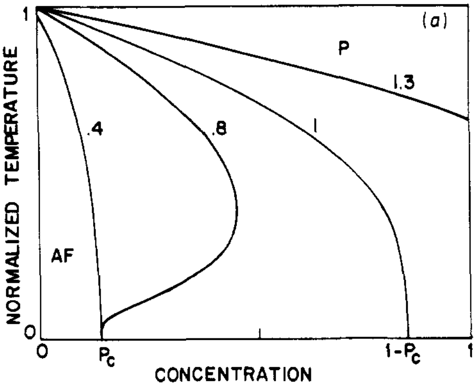
$$\alpha = \tfrac{1}{2}K \ln\{(\sqrt{2}\pm 1)[(1+\eta X)/(1+\eta Y)]\} \tag{9}$$

with $\eta$ given by equation (5).

### 2.2. *The phase diagrams*

In figure 2 we show the normalised temperature against the concentration phase diagram for $M = 1, 2$ (monomer and dimer), for several values of the competition parameter $\alpha$. The trimer decoration has a behaviour similar to the dimer case. The appearance of both the ferromagnetic (high concentrations) and antiferromagnetic (low concentrations) phases occur for values of $0 < \alpha < 1$ for all cases (see figure 2($a$) of [17]), but in figure 2 to avoid misunderstanding we show only the antiferromagnetic phase. We observe that a re-entrance with two transition temperatures occurs for all value of $M$ for concentrations $p > p_c$, where $p_c = (1-\sqrt{2}/2)2$. In figure 2($b$) a double re-entrance, with three transition temperatures, occurs for $M > 2$, $p > \frac{1}{2}$, and for certain values of the coupling parameters of the decorating variables with the spins of the lattice. The multiple re-entrance is very sensitive to the values of these parameters. For example, if we change $\lambda$ from 0.8 to 0.5, keeping all other parameters at the same values, the double re-entrance shown in figure 2 disappears.

In figure 3 we present the reduced temperature against the competition parameter phase diagram for the cases $M = 1, 2$ and for several values of the concentration (the case $M = 3$ is similar to the case $M = 2$). The abrupt appearance at $p_c$ of the single re-entrant phase is evident from these figures. The behaviour of the $p = 0.14$ curve is completely different from that of $p = 0.15$. For $M = 1$ the ferromagnetic phase disappears for $\alpha > 1$ (figure 3($a$)). For $M = 2$ the ferromagnetic ground state becomes unstable at $\alpha_c = \gamma + 2\lambda$. For $\alpha \geqslant \alpha_c$ the sequence of transitions from the ground state



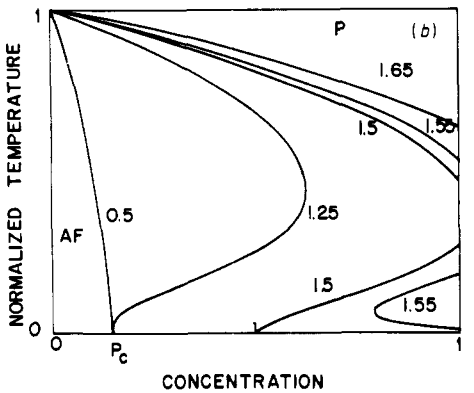




**Figure 2.** Normalised critical temperature against concentration phase diagram for the antiferromagnetic-paramagnetic boundaries of random-decorated model: (*a*) $M = 1$ monomer decoration for $\alpha = 0.4$, 0.8, 1.0 and 1.3; (*b*) $M = 2$ dimer decoration with $\lambda = 0.8$ and $\gamma = -0.1$ for $\alpha = 0.5$, 1.25, 1.5, 1.55 and 1.65.

is antiferro-para-antiferro-para. For values smaller than $\alpha_c$, and within a certain range of concentrations, the transition sequence ferro-para-antiferro-para. Moreover, as long as we keep the antiferromagnetic coupling between the spins at the lattice sites $\alpha = 0$ is the other limit of the ferromagnetic phase.

## 3. Axial decorated model

### 3.1. Model description and general solution

In this section we consider the two-dimensional square lattice Ising model decorated in one direction (axial decoration) with classical variables, as in the previous section. In this case no randomness is present. The Hamiltonian of the system is given by

$$H = J_0 \Sigma\, \sigma_{l,m}\sigma_{l,m+1} + \alpha J \,\Sigma\, \sigma_{l,m}\sigma_{l+1,m} - \Sigma\, (\Sigma\, JS_{i,l}(\lambda_i\sigma_{l,m} + \lambda'_i\sigma_{l+1,m})) - \Sigma\, J_i S_{i,l} S_{i+1,l}. \qquad (10)$$

The spins on the lattice sites ($\sigma_{l,m}$) interact with coupling constant $J_0 = rJ$ along the $y$ direction and with coupling constant $-\alpha J$ along the $x$ direction. The decorating

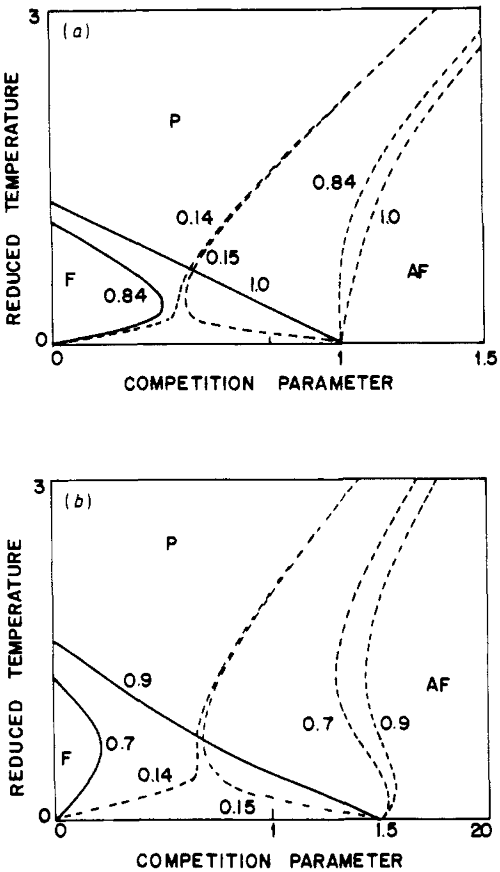



**Figure 3.** Reduced critical temperature against competition parameter phase diagram of the random-decorated model: ($a$) $M=1$ monomer decoration for $p=0.14$, 0.15, 0.84 and 1.0; ($b$) $M=2$ dimer decoration for $p=0.14$, 0.15, 0.7 and 0.9 with $\lambda=0.8$ and $\gamma=0.1$ (full curve) ferromagnetic (F) and (broken curve) antiferromagnetic (AF) boundaries.

variables $S_{i,l}$ are located in all bonds along the $x$ direction and they interact with site spins with coupling constants $\lambda_i J$ and $\lambda'_i J$ for arbitrary $\lambda_i$ and $\lambda'_i$. The decorating variables interact with each other with coupling constant $J_i=\gamma_i J$. The sum over $i$ runs over all decorating bond variables. Figure 1($c$) shows a portion of the lattice.

By a similar procedure, as before, we obtain after performing the trace over the decorating variables, an Ising model on a square lattice with an interaction $K_0=\beta J_0$ along the $y$ direction with an effective interaction along the $x$ direction, given by

$$K_{\text{eff}}=-\alpha K+\ln(X/Y)/2 \tag{11}$$

where $X$ and $Y$ are given in table 1, and $K=\beta J$.

The transition temperatures are obtained from the solutions of the equation,

$$\sinh(2K_0)\sinh(2K_{\text{eff}})=\pm 1 \tag{12}$$

where the $+(-)$ stands for the ferromagnetic (antiferromagnetic) phase.

The phase diagram temperature against the competition parameter $\alpha$, which measures the ratio between the intersites antiferromagnetic coupling and the ferromagnetic one between site and bond spins ($\lambda_i=\lambda'_i=1$), is obtained analytically from the previous equation, that is

$$\alpha = \frac{1}{2K} \ln\left[\frac{X}{Y}\left(\frac{\operatorname{sign}(K_0 \cosh(2K_0 \pm 1)}{\sinh(2K_0)}\right)\right]. \tag{13}$$

The specific heat is obtained from the free energy as

$$C_H = -k_B\beta^2\partial^2/\partial\beta^2(\beta F)|_H. \tag{14}$$

The free energy of the model is

$$F = -(1/k_B T)\ln(D\lambda_{max}) \tag{15}$$

where $D = \sqrt{XY}$ and $\lambda_{max}$ is the largest eigenvalue of the partition function of this asymmetric lattice with coupling constants $J_0$ and $J_{eff}$,

$$\ln(\lambda_{max}) = \tfrac{1}{2}\ln|2\sinh(2K_0)| + \frac{1}{2\pi}\int_0^{\pi} \cosh^{-1}\Omega \, d\omega \tag{16}$$

with

$$\Omega = \cosh(2K_{eff})\cosh(2K^*) - \sinh(2K_{eff})\sinh(2K^*)\cos\omega \tag{17}$$

and

$$K^* = \tfrac{1}{2}\ln|\coth K_0|. \tag{18}$$

### 3.2. *Phase diagram $T_c x\alpha$ and specific heat*

In figure 4 we present the phase diagram for $M = 1, 2, 3$. The system shows three phases: paramagnetic (P), ferromagnetic (F) and a mixed (M) phase. The ground state of the mixed phase is characterised by an antiferromagnetic configuration for the site spins. Figure 4($a$) shows the single re-entrance for the monomer decoration. The dimer decoration can exhibit a single re-entrance as well as a double re-entrance, as shown in figure ($b$). The multiple re-entrance (three or more) is achieved by decorating the bonds with more than two variables as presented in figure 4($c$) for the $M = 3$ case. In these figures, a sequence of five transitions occurs for values of $\alpha$ in the range of 0.175 for the dimer and 0.345 for the trimer, alternating between the mixed, paramagnetic and ferromagnetic phases. Note that for $\alpha = 0.315$ we have a sequence of seven transitions in the $M = 3$ case as shown in figure 4($c$). These transitions will be more evident in the specific heat calculations. Again, by comparing the three curves in figure 4($b$) of the $M = 2$ case, and also the curves in figure 4($c$) for the $M = 3$ case, one can see how sensitive the phase diagram is to the values of the exchange couplings of the decorating variables. We point out that the sign of the coupling constant on the undecorated direction is irrelevant for the transition lines. Only the configurations of the site spins will be affected by that sign, but the shape of the transition lines remain unchanged. We illustrated this by presenting the phase diagram with $J_0 > 0$ for the dimer case and with $J_0 < 0$ for the trimer case. The specific heat for the dimer and trimer decoration, for $\alpha = 0.175$ and $\alpha = 0.345$, respectively (these points are indicated in figures 4($b$) and 4($c$)) are presented in figure 5. The specific heat results show clearly the sequence of phase transitions. For $T \approx 0.07$ the specific heat for the dimer decoration, shown in figure 5($a$), has a double peak indicating the two transitions F–P and P–F. This is not clear from figure 5($a$). However, in figure 5($b$) we present a detail of this double peak on the specific heat appearing at $T \approx 0.07$. Similarly, for the trimer decoration a double peak on the specific heat occurs for temperatures close

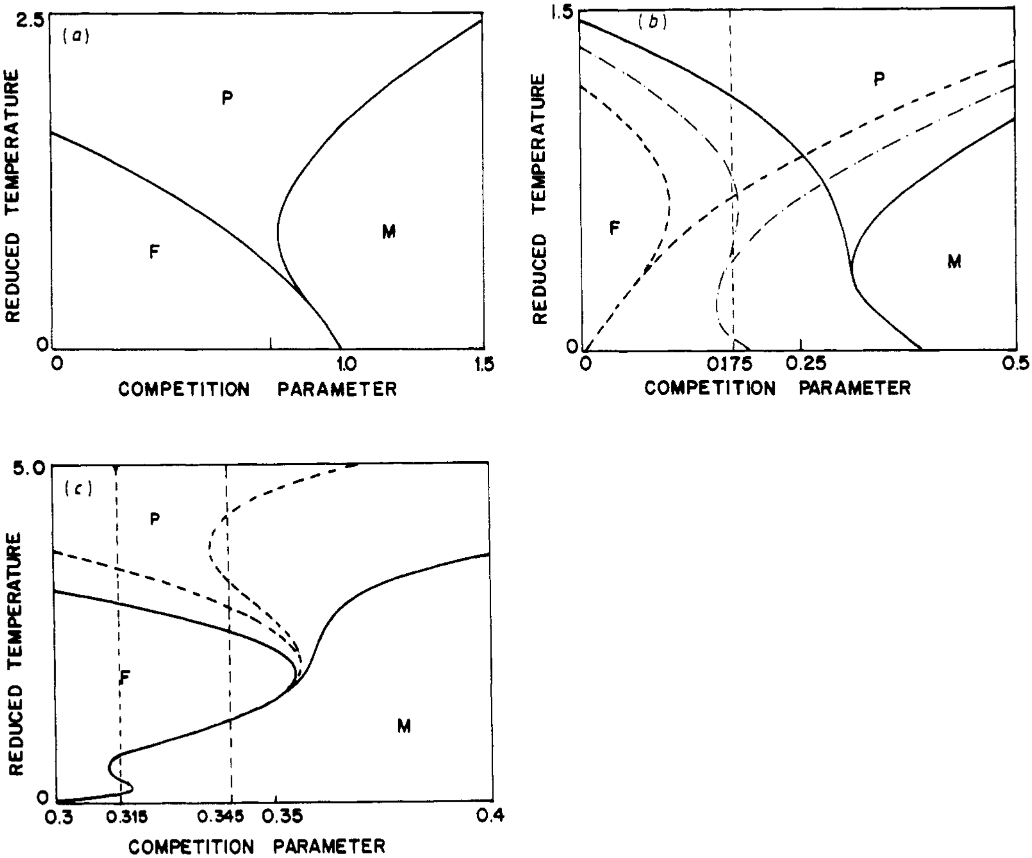



**Figure 4.** Reduced critical temperature against competition parameter phase diagram of the pure axial-decorated model. ($a$) $M=1$ monomer decoration. ($b$) $M=2$ dimer decoration with $\lambda=1.0$ and $\gamma=-1.6$ (full curve), $\gamma=-1.8$ (chain curve) and $\gamma=2.0$ (broken curve). ($c$) $M=3$ trimer decoration with $\gamma_1=-\gamma_2=0.1$, $\gamma_3=-2.0$, $\lambda=1.0$, $\lambda_1=0.3$ and $\lambda_2=0.5$ for $r=-7.0$ (full curve) and $r=-10$ (broken curve).

to $T\approx 1.2$ (see figure 5($c$)). On the other hand, the peaks in the low-temperature regime seem to be associated with the onset of local ordering within the decorating complex in the presence of two possible ordering configurations (↑↑↑↑↑↑ or ↑↓↑↓↑↓) of the site spins on the extremes of the decorated bond, as a function of the competing parameter $\alpha$. This interpretation explains the slight shift in the temperature in which this peak occurs in the two different site configurations. This, we have obtained by analysing, in detail, the specific heat in the vicinity of $T\approx 0$. This peak occurs at two different, very low temperatures, fixed for all values of $\alpha$ separated by the value $\alpha\approx 0.31$.

## 4. Concluding remarks

We have studied a square-lattice Ising model decorated with dimers or trimers with competing interactions in two different scenarios. In the first case we considered a random annealed isotropically decorated lattice and in the second case we studied the pure axial-decorated model in which only the bonds in one direction are decorated.

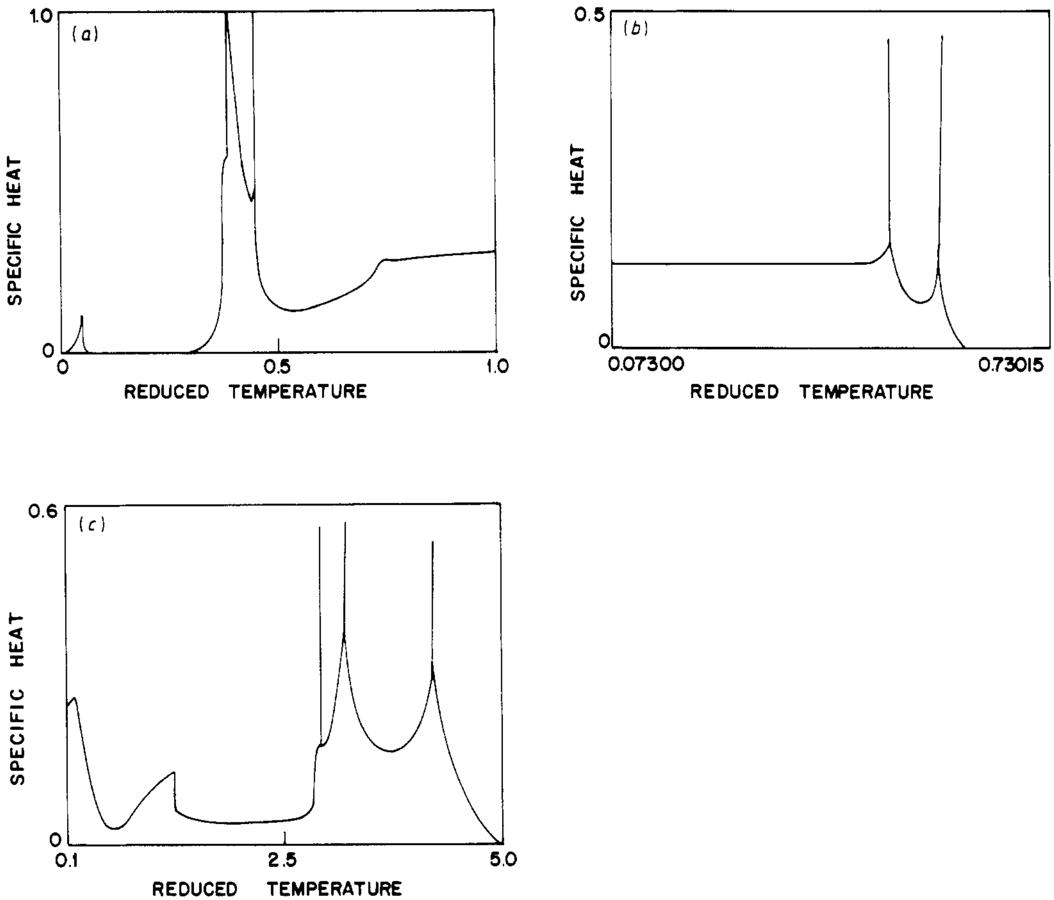



**Figure 5.** Specific heat against reduced temperature of the pure axial-decorated model: ($a$) for $M=2$ dimer decoration with $\alpha=0.175$, $\gamma=-1.8$ and $\lambda=1.0$, ($b$) reduced scale of figure 5($a$) pointing out the double peak around $T\simeq 0.07$; ($c$) for $M=3$ trimer decoration with $\alpha=0.34$ and with the same parameters of figure 4($c$). At $T\simeq 1.2$ a double peak occurs in the same fashion as shown in figure 5($b$). In all figures the vertical lines are guides for the eyes.

Our intention was to show that if we increase the complexity of the decorating system we can obtain, for well chosen ranges of the competing parameters, multiple re-entrant behaviour for the ordered phases of the site spins. We succeed in showing this, with the most remarkable results appearing for the axial decorated cases. It is possible that even more complex behaviour can be obtained for choices of parameter sets different from the ones we have used in this paper. We made no attempt to sweep all the parameter space of the different models.

In general, the appearance of the multiple re-entrant behaviour when we increase the complexity of the decorating system can be explained in terms of the minimisation of the free energy by the increase of the entropy of the frustrated system [18]. The minimum of the internal energy defines the ground state ($T=0$). As the temperature increases the gain in the entropy, the negative contribution to the free energy ($-TS$), can exceed the loss of the free energy due to the internal energy term. In particular, let us consider the monomer decorated case, in which the decorating variable interacts with the sum of the site spins, while the site spins interact with each other with an antiferromagnetic coupling $-\alpha J$, i.e., the decorated-bond Hamiltonian is $H_b = JS(\sigma_1+\sigma_2)+\alpha J\sigma_1\sigma_2$. If the value of $\alpha$ is smaller, but close to 1 it should be energetically more advantageous for the systems, as the temperature is raised from zero, to assume the antiferromagnetic configuration, in which case the second parcel of the Hamiltonian is minimised and the spin $S$ in the first parcel becomes free to contribute for the entropy of the system. For the disordered model an additional mechanism could come into play to account for re-entrances in the phase diagram, as discussed in [18]. At a given concentration of diluted bonds for which the ground state is paramagnetic the increasing of temperature can induce the bonds to move resulting in an ordered phase in order to minimise the free energy.

In summary, for the competing random model we obtained a single re-entrance for all decorating variables for concentrations $p>(1-\sqrt{2}/2)/2$, while double re-entrances appeared for two or more decorating variables and for $p>\frac{1}{2}$. The ferromagnetic phase becomes unstable for $\alpha>1$ in the monomer decoration, and for $\alpha>\gamma+2\lambda$ for the two other types of decoration. In the axial-decorating model we showed the multiple re-entrance phenomena with the appearance of a sequence of several transitions. Also, the results show that the sign of the coupling constants in the undecorated direction affects the configuration of the site spins, but keeps unchanged the transition lines. Finally, we remark that the results presented in this paper are all exact, once they rely on the pure 2D Ising square-lattice Onsager solution.

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